Suppose, if possible, that ρ is real. Then if $\phi(t)$ is defined, in terms of the solution $x(t)$ occurring in (3), by

$$x(t) = |\rho|^{t/\pi}\phi(t),$$

it is seen that

$$\phi(t + \pi) = \epsilon\,\phi(t), \text{ where } \epsilon = sgn\rho = \pm 1.$$

Hence $\phi(t)$ has 2π as a period. Since $f(t)$ is real, the real and imaginary parts of $x(t)$ are solutions of (2). Consequently, (2) possesses a non-trivial solution of the form

$$x(t) = |\rho|^{t/\pi}\psi(t) \not\equiv 0, \tag{4}$$

where $\psi(t)$ is real and of period 2π.

Since the solution (4) is real, it follows from the inequalities (1), from Sturm's comparison theorem, and from the assumption that $f(t)$ is not constant, that

$$2n < N < 2(n + 1), \tag{5}$$

where N denotes the number of zeros of the solution (4) on a half-open interval of length 2π, say on $0 \leqq t < 2\pi$.

On the other hand, (4) shows that $x(t)$ and $\psi(t)$ have the same zeros. Since $\psi(t)$ is periodic, the number N of zeros of $\psi(t)$ on a period, $0 \leqq t < 2\pi$, is even. Since this contradicts (5), the proof is complete.

[1] Borg, G., *Ark. f. Matemat., Astr. o. Fysik*, **31**, No. 1, p. 28.

[2] Strutt, M. J. O., *Lamésche, Mathieusche und Verwandte Funktionen in Physik und Technik*, Berlin, 1932, pp. 24 and 40.

ON SOME EXPONENTIAL SUMS

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It seems to have been known for some time[1] that there is a connection between various types of exponential sums, occurring in number-theory, and the so-called Riemann hypothesis in function-fields. However, as I was unable to find in the literature a precise statement for this relationship, I shall indicate it here, and derive from it precise estimates for such sums, including the Kloosterman sums.

Let k be a finite field of q elements; consider the field $k(t)$ of rational functions in one transcendental element t, with coefficients in k; geometrically, this is the function-field, over the ground-field k, of a projective straight line. On that straight line, we consider divisors, i.e., formal sums of points with integral (positive or negative) coefficients; and we limit ourselves, once for all, to divisors which are rational over k, i.e., such that conjugate points over k have the same coefficient. A divisor is called finite if it does not contain the point at infinity with a non-zero coefficient. Except for the notation, finite positive divisors are essentially the same as ideals in the ring $k[t]$; to every such divisor $\mathfrak{a}$, we attach the polynomial $P_{\mathfrak{a}}(t) = t^n + a_1 t^{n-1} + \ldots + a_n$ which generates the corresponding ideal, i.e., whose zeros are the points in $\mathfrak{a}$, with multiplicities respectively equal to their coefficients in $\mathfrak{a}$; as $\mathfrak{a}$ is assumed to be rational over k, $P_{\mathfrak{a}}(t)$ has its coefficients in k; and n is the degree of $\mathfrak{a}$. Every finite divisor $\mathfrak{m}$ can be written as $\mathfrak{m} = \mathfrak{a} - \mathfrak{b}$, where $\mathfrak{a}$, $\mathfrak{b}$ are finite positive divisors; to $\mathfrak{m}$, we attach the function $R_{\mathfrak{m}}(t) = P_{\mathfrak{a}}(t)/P_{\mathfrak{b}}(t)$; we have $\mathfrak{m} \sim 0$ if and only if $\mathfrak{a}$ and $\mathfrak{b}$, i.e., $P_{\mathfrak{a}}(t)$ and $P_{\mathfrak{b}}(t)$, are of the same degree, and then there is one and only one function in $k(t)$ having $\mathfrak{m}$ as its divisor and taking the value 1 at infinity, viz., $R_{\mathfrak{m}}(t)$ itself.

Let χ be a character of the multiplicative group k^* of the non-zero elements in k. Let $\mathfrak{d}$ be a finite divisor, consisting of the points ξ_ν with the coefficients a_ν; if $R(t)$ is in $k(t)$, we shall write $R(\mathfrak{d}) = \Pi_\nu R(\xi_\nu)^{a_\nu}$ whenever none of the $R(\xi_\nu)$ is 0 or ∞; as $\mathfrak{d}$ is rational over k, $R(\mathfrak{d})$ is in k. We shall assume that no a_ν is a multiple of the order of χ.

Furthermore, let ω be a character of the multiplicative group of power series in an indeterminate T with coefficients in k; we assume that ω has the value 1 for every series reduced to a monomial cT^m. According to the usual definition, we say that ω has the conductor (T^N) if it has the value 1 for every power-series which is $\equiv 1$ mod. T^N, and if N is the smallest integer with that property. Then the values of ω are p^s-th roots of unity, if p is the characteristic of k, and s is such that $p^s \geqslant N$. We shall write, for $x \in k$, $\lambda(x) = \omega(1 - xT)$. To every function $R(t)$ in $k(t)$, we can attach a power-series $R(1/T)$, arising from the expansion of the rational function $R(1/T)$ according to increasing powers of T; this is no other than the usual expansion of $R(t)$ at infinity. Then $\omega[R(1/T)]$ is defined; in particular, as $\omega(T) = 1$, we have $\omega[(1 - xT)/T] = \lambda(x)$. Now, for every finite divisor $\mathfrak{m}$ with no point in common with $\mathfrak{d}$, we write

$$\varphi(\mathfrak{m}) = \omega[R_{\mathfrak{m}}(1/T)]\cdot\chi[R_{\mathfrak{m}}(\mathfrak{d})]. \tag{1}$$

This depends multiplicatively upon $\mathfrak{m}$, i.e., $\varphi(\mathfrak{m} + \mathfrak{n}) = \varphi(\mathfrak{m})\varphi(\mathfrak{n})$. Furthermore, if $\mathfrak{m} \sim 0$, and if there is a function $R(t)$ in $k(t)$, having $\mathfrak{m}$ as its divisor, taking the value 1 at every point ξ_ν, and such that $R(1/T) \equiv 1$ mod. T^N, we have $\varphi(\mathfrak{m}) = 1$; for $R(t)$ can then be no other than $R_{\mathfrak{m}}(t)$. According to well-known definitions, this shows that $\varphi(\mathfrak{m})$ is an Abelian character over the field $k(t)$, whose conductor consists of the point at infinity with the coefficient N, and of the points ξ_ν with the coefficient 1; if d is the number of points ξ_ν in $\mathfrak{b}$, the degree of that conductor is therefore $N + d$; hence, by a known theorem,[2] the L-series belonging to this character is a polynomial of degree $N + d - 2$; calling α_i its roots, we have thus

$$\sum_{\mathfrak{a}} \varphi(\mathfrak{a})\cdot u^{n(\mathfrak{a})} = \prod_{i=1}^{N+\alpha-2} (1 - \alpha_i u), \tag{2}$$

where the sum in the left-hand side is extended to all finite positive divisors $\mathfrak{a}$ with no point in common with $\mathfrak{b}$, and where $n(\mathfrak{a})$ is the degree of $\mathfrak{a}$. Writing that the terms in u are equal on both sides, we get

$$\sum \varphi(\mathfrak{a}) = -\sum_i \alpha_i, \tag{3}$$

where the sum in the left-hand side is now extended only to the finite positive divisors of degree 1. These are in one-to-one correspondence with the polynomials $P_{\mathfrak{a}}(t) = t - x$, with $x \in k$. For such a divisor, we have

$$R_{\mathfrak{a}}(1/T) = P_{\mathfrak{a}}(1/T) = (1 - xT)/T,$$

hence $\omega[R_{\mathfrak{a}}(1/T)] = \lambda(x)$, and also

$$R_{\mathfrak{a}}(\mathfrak{b}) = \prod_\nu (\xi_\nu - x)^{a_\nu} = (-1)^a R_{\mathfrak{b}}(x),$$

with $a = \Sigma a_\nu$. Then (3) can be written as

$$\sum \lambda(x)\chi[R_{\mathfrak{b}}(x)] = (-1)^{a+1}\sum \alpha_i, \tag{4}$$

where the sum in the left-hand side is over all the elements x of k, other than the ξ_ν if any of these is in k. We may extend that sum to all elements x of k by agreeing that $\chi(0) = \chi(\infty) = 0$.

By class-field theory, the character $\varphi(\mathfrak{m})$ belongs to an Abelian extension of $k(t)$, and its L-series divides the zeta-function of that extension. Therefore, by the Riemann hypothesis,[3] all the α_i have the absolute value $\sqrt{q}$, hence

$$\left|\sum \lambda(x)\chi[R_{\mathfrak{b}}(x)]\right| \leqslant (N + d - 2)\sqrt{q}. \tag{5}$$

For instance, we can define a character ω, of conductor (T^2), by putting, for every series of constant term 1:

$$\omega(1 + x_1T + x_2T^2 + \ldots) = -\psi(x_1),$$

where ψ is a character of the additive group of k, not everywhere equal to 1. This gives

$$\left|\sum \psi(x)\chi[R_{\mathfrak{b}}(x)]\right| \leqslant d\sqrt{q}.$$

If the characteristic p of k is not 2, we have $d = 2$ for $R_{\mathfrak{b}}(t) = t^2 - a$, $a \neq 0$; hence, in that case,

$$|\sum\psi(x)\chi(x^2 - a)| \leqslant 2\sqrt{q}.$$

If, in this, we take for χ the character of k^* of order 2 (equal to 1 for squares, and to -1 for non-squares, in k^*), an elementary transformation[4] shows that the sum in the left-hand side is identical with the so-called Kloosterman sum $\Sigma\psi(cx + dx^{-1})$, for $4cd = a;$ hence

$$|\sum_{x\neq 0}\psi\,(cx + dx^{-1})| \leqslant 2\sqrt{q},$$

and, in the case of a prime field of p elements, with $p \neq 2$:

$$|\sum_{x=1}^{p-1} e^{2\pi i/p\,(cx + d/x)}| \leqslant 2\sqrt{p}.$$

Furthermore, it is easily seen, e.g., by induction on n, that, if $F(x)$ is[4] any polynomial in x of degree n, with coefficients in k, such that $F(0) = 0$, there exists at least one character ω, of conductor (T^N) for some $N \leqslant n + 1$, such that, with the above notations, $\lambda(x) = \psi[F(x)]$. Then (5) gives:

$$|\sum\psi[F(x)]\chi[R_b(x)]| \leqslant (n + d - 1)\sqrt{q}.$$

[1] Cf., for example, H. Rademacher's excellent report on analytic number theory, *Bull. A. M. S.*, **48**, 379–401 (1942).

[2] Weissinger, J., *Hamb. Abhandl.*, **12**, 115–126 (1938).

[3] Weil, A., *Pub. Inst. Math. Strasbourg* (*N.S.*, no. 2), pp. 1–85 (1948).

[4] Davenport, H., *Crelles J.*, **169**, 158–176 (1933); cf. in particular Th. 5, p. 172.

ON SPACES WITH VANISHING LOW-DIMENSIONAL HOMOTOPY GROUPS

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This note contains an investigation of the relationships between some of the homotopy and homology groups of an $(n - 1)$-connected space (i.e., a pathwise connected topological space whose homotopy groups of dimensions $< n$ all vanish).

Let X be an $(n - 1)$-connected space, and let A be a set of generators for the nth homotopy group, $\pi_n(X)$. For each $\alpha \in A$, let E_α^{n+1} be an $(n + 1)$-cell with boundary S_α^n; let y_α be a fixed reference point of S_α^n; x_0 a fixed reference point in X; and let f_α: $(S_\alpha^n, y_\alpha) \to (X, x_0)$ be a mapping representing the element $\alpha \in \pi_n(X)$. Suppose that $\bigcup_{\alpha\in A} E_\alpha^{n+1}$ is topologized so that the cells E_α^{n+1} are mutually separated and let E be the topological space obtained from $\bigcup_{\alpha\in A} E_\alpha^{n+1}$ by identifying all the points y_α to a single